Indian Statistical Institute, Bangalore Centre
M.Math. (I Year) : 2013-2014
Semester I : Semestral Examination
Measure Theoretic Probability

11.11.2013 Time: 3 hours. Maximum Marks : 100

Note: The paper carries 105 marks. Any score above 100 will be taken as 100. Notation and terminology are understood to be as used in class. State clearly the results you are using in your answers.

1. (20 marks) Let $(\Omega, \mathcal{B}, \mu)$ be a measure space. Let $A_n \in \mathcal{B}, n = 1, 2, \cdots$ Define

$$\limsup A_n = \{\omega \in \Omega : \omega \in A_n \ \text{ for infinitely many } \ n \geq 1\}.$$

Suppose that $\sum_{n=1}^{\infty} \mu(A_n) < \infty$. Show that $\limsup A_n \in \mathcal{B}$, and that $\mu(\limsup A_n) = 0$.

2. (20 marks) Let $f_n, n \geq 1, f$ be nonnegative integrable functions on a $\sigma-$finite measure space $(\Omega, \mathcal{B}, \mu)$. Suppose that $\int_\Omega f_n(\omega) d\mu(\omega) = \int_\Omega f(\omega) d\mu(\omega)$ for all n, and that $f_n \to f$ as $n \to \infty$, $\mu - a.e.$ Show that

$$\sup_{A \in \mathcal{B}} \mid \int_A f(\omega) d\mu(\omega) - \int_A f_n(\omega) d\mu(\omega) \mid \to 0,$$

as $n \to \infty$. (*Hint:* First show that l.h.s. of the above is dominated by $\int_\Omega |f(\omega) - f_n(\omega)| d\mu(\omega)$. Also note that $0 \leq (f - f_n)^+ \leq f$.)

3. $(15 + 10 = 25$ marks) (i) Let λ be a probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, which is absolutely continuous w.r.t. the Lebesgue measure with probability density function $f(\cdot)$. Let μ be any probability measure on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Show that $\lambda * \mu$ is absolutely continuous w.r.t. the Lebesgue measure, with probability density function given by

$$g(x) = \int_{\mathbb{R}} f(x - y) d\mu(y), \ x \in \mathbb{R}.$$

(ii) X, Y are independent real valued random variables such that X has the standard normal distribution, and Y has geometric distribution with parameter $\frac{1}{2}$. Find the probability density function of $X + Y$.

4. (10 + 5 =15 marks) (i) Let μ_a denote the distribution of a random variable having exponential distribution with parameter $a > 0$. Find the characteristic function of μ_a.

   (ii) Let $a_n \to a$, where $a_n, a > 0$. Using (i) or otherwise show that $\mu_{a_n} \Rightarrow \mu_a$, as $n \to \infty$.

5. (25 marks) For $a \geq 0, n = 1, 2, \cdots$ put

$$g(n,a) = e^{-n} \sum_{k \leq na} \frac{n^k}{k!}.$$

   Using the central limit theorem, find $\lim_{n\to\infty} g(n,a)$. (*Hint:* Consider the cases $a < 1, a = 1, a > 1$ separately.)